Journal of Statistical Physics, Vol. 108, Nos. 5/6, September 2002 (© 2002)

Leading Branches of the Transfer-Matrix Spectrum for Lattice Spin Systems (Quasi-Particles of Different Species)

R. Minlos[1] and E. Zhizhina[1]

Received September 13, 2001; accepted March 14, 2002

For lattice systems under high temperatures T with compact or finite spin we construct three invariant subspaces of the transfer-matrix, which can be interpreted as the spaces of states for quasi-particles of two different species and the space of states for two particles of the first species. We formulate a condition on *a priori* distribution guaranteeing that the spectrum of the transfer-matrix on these subspaces are not overlapping.

KEY WORDS: Lattice Gibbs fields; invariant subspaces of the transfer-matrix; cluster estimates; transfer-matrix spectrum.

1. INTRODUCTION AND MAIN RESULTS

We consider a Gibbs spin field on the lattice $Z^{d+1} = Z^d \times Z$ ("space" × "time") with a nearest neighbor interaction. The spin takes values from a compact set $S \subset R^1$, and let ν be *a priori* spin probability distribution on S. The stochastic operator $\mathcal{T}$ of the corresponding Markov chain is known as the transfer-matrix. It is a self-adjoint operator in a corresponding Hilbert space $\mathcal{H}$. The goal of the paper is to study the structure of the leading invariant subspaces of the transfer-matrix and to find the corresponding upper branches of the spectrum of the transfer-matrix (or what is the same, the lower spectrum branches of the Hamiltonian $H = -\frac{1}{2} \ln \mathcal{T}^2$ associated with a lattice quantum system) under high temperatures $T = \beta^{-1} \gg 1$.

Our constructions are the following. An invariant subspace $\mathcal{H}_1 \subset \mathcal{H}$ with the corresponding transfer-matrix spectrum of the order β is found

Dedicated to D. Ruelle and Ya. Sinai.

[1] Institute for Information Transmission Problems, Bolshoy Karetny per. 19, 101447 Moscow, Russia; e-mail: ejj@iitp.ru

0022-4715/02/0900-0885/0

first. The subspace $\mathscr{H}_1$ has the following structure. There exists an orthonormal basis $\{u_x, x \in Z^d\}$ in $\mathscr{H}_1$. The elements of the basis are marked by points of the lattice Z^d, and $\mathscr{U}_s u_x = u_{x+s}$ for any $x, s \in Z^d$, where $\{\mathscr{U}_s, s \in Z^d\}$ is the unitary group of the space translations acting in $\mathscr{H}$. Hence the subspace $\mathscr{H}_1$ is cyclic under the unitary group $\{\mathscr{U}_s, s \in Z^d\}$, and it is similar by the structure to a space of quasi-particle states in physics. For this reason we call $\mathscr{H}_1$ as the one-particle invariant subspace of the transfer-matrix. Let us note that after the Fourier transform

$$u_x \to \exp\{i(\lambda, x)\}, \qquad x \in Z^d, \quad \lambda \in T^d$$

we get the unitary transformation of the space $\mathscr{H}_1$ to the Hilbert space $L_2(T^d)$ of functions defined on a d-dimensional torus T^d. Here T^d is the space of quasi-momentum of our "particle." In this case the transfer-matrix is unitary equivalent to the operator of the multiplication by a function $c(\lambda)$, $\lambda \in T^d$, and the energy of the "particle" is determined by $-\frac{1}{2} \ln c^2(\lambda)$.

Next we construct an invariant subspace $\mathscr{H}_2$, such that the transfer-matrix spectrum on $\mathscr{H}_2$ has the order β^2 and in the orthogonal complement to the sum of subspaces $\mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2$, where $\mathscr{H}_0 = \{\mathbf{1}\}$ is the space of constants, the transfer-matrix spectrum has the order β^3. Therefore our constructions imply that the whole spectrum of $\mathscr{T}$ of the order β^2 is the same as the spectrum of $\mathscr{T}$ restricted to $\mathscr{H}_2$.

Then we prove, that under a certain condition on *a priori* distribution ν (non-even, in general) on S with $\#S > 2$ the invariant subspace $\mathscr{H}_2$ could be decomposed into an orthogonal sum of subspaces

$$\mathscr{H}_2 = \mathscr{H}_2^{(1)} \oplus \mathscr{H}_2^{(2)}, \tag{1}$$

and the spectra of $\mathscr{T}$ on these subspaces are not overlapping:

$$\sigma(\mathscr{T}|_{\mathscr{H}_2^{(1)}}) \cap \sigma(\mathscr{T}|_{\mathscr{H}_2^{(2)}}) = \varnothing. \tag{2}$$

Both subspaces in (1) are invariant with respect to $\mathscr{T}$ and the unitary group of the space translations $\{\mathscr{U}_s, s \in Z^d\}$. The subspace $\mathscr{H}_2^{(2)}$ has a so-called two-particle structure, it describes states of two particles of the first species, associated with the subspace $\mathscr{H}_1$. The subspace $\mathscr{H}_2^{(1)}$ has a structure, which is completely similar to the structure of the one-particle subspace $\mathscr{H}_1$. The decomposition (1) together with the spectral analysis of the transfer-matrix on the subspaces $\mathscr{H}_2^{(j)}$ $(j = 1, 2)$ are the main results of the paper.

As follows from our constructions below, the spectrum branch $\sigma(\mathscr{T}|_{\mathscr{H}_2^{(1)}})$ can be found both above and below to the spectrum branch $\sigma(\mathscr{T}|_{\mathscr{H}_2^{(2)}})$. This fact together with the internal one-particle structure of the subspace $\mathscr{H}_2^{(1)}$ is the main reason, why we propose to interpret both of $\mathscr{H}_1$ and $\mathscr{H}_2^{(1)}$ as the spaces of states for different species of quasi-particles. Let us note that the interpretation of $\mathscr{H}_2^{(1)}$ carries a purely terminology convention. Many authors propose to consider bound states of two, three, etc. particles as states of a new single particle, and our results confirm this standpoint.

This paper is inspired by refs. 1 and 2, where the similar results have been obtained using the Bethe–Salpeter equation method. However, the spectrum of $\mathscr{T}|_{\mathscr{H}_2^{(1)}}$ was found in refs. 1 and 2 only in the case, when it lies strictly above to the spectrum of $\mathscr{T}|_{\mathscr{H}_2^{(2)}}$. We don't impose here this restriction, but we need the compactness of the spin space S (since cluster estimates, obtained in refs. 3 and 4, hold true only in the case of a compact spin space).

Let us remark, that in the general case the structure of the two-particle invariant subspace $\mathscr{H}_2^{(2)}$ can be complex. It can contain by itself some additional two-particle bound states, associated with values of the quasi-momentum running some regions of the torus T^d, see, for example, ref. 5. Here we don't consider the problem of the existence of these two-particle bound states of the operator $\mathscr{T}|_{\mathscr{H}_2^{(2)}}$. This problem requires a more detailed spectral analysis of the transfer-matrix. A partial case, when $S = \{-1, 0, 1\}$ and v is a symmetrical distribution on S, was studied in ref. 6, where the authors found the spectrum of $\mathscr{T}|_{\mathscr{H}_2}$ completely, including two-particle bound states.

The problem of the existence of two-particle bound states has been considered also in quantum field theories. This problem was studied in weakly coupled $\mathscr{P}(\varphi)_2$ quantum field models by many authors, see, for instance, refs. 7–9. The results as well as the methods of the papers of refs. 7–9 are different from the results and approaches of this work. Continuous quantum field models are considered as perturbations of the Gaussian field. The Gaussian field is associated with a "free system," and the spectrum of the Hamiltonian of the quantum system appears to be close by the structure to the spectrum of the Hamiltonian of the "free system." In our paper we study perturbations of an independent (non-interacting) field, resulting in a different structure for the spectrum of the transfer-matrix.

We now describe the model and state results. Let (S, v) be a single spin space. Here $S \subset R$ is a compact subset of R and v is *a priori* probability distribution on S. We suppose that

$$\operatorname{supp} v = S, \qquad \text{and} \qquad \#S > 2. \tag{3}$$

Denote by $\Omega = S^{Z^{d+1}}$ the space of configurations

$$\sigma = \{\sigma(x), x \in Z^{d+1}\}, \qquad \sigma(x) \in S,$$

and let

$$\mu_0 = \nu^{Z^{d+1}}$$

be the free measure on Ω. The formal Hamiltonian of the system has the following form:

$$H(\sigma) = -\sum_{\substack{x, x' \in Z^d \\ |x-x'|=1}} \sigma(x) \cdot \sigma(x').$$

We consider here the high temperatures case, when $\beta = T^{-1} \ll 1$. The corresponding Gibbs measure on Ω is denoted by μ_β.

Let e_{d+1} be the unit vector along the "time" direction, and

$$Y_k = \{x = (x_1,..., x_{d+1}) \in Z^{d+1} : x_{d+1} = k\} \subset Z^{d+1}$$

is the kth "time slice," $k \in Z$. Every configuration $\sigma \in \Omega$ can be written as a sequence

$$\sigma = \{..., \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2,...\}, \tag{4}$$

where $\sigma_k = \sigma|_{Y_k} \in S^{Z^d} \equiv \Omega_0$. In addition μ_β is the distribution of a stationary Markov chain (4) with the state space Ω_0 and the invariant measure $\pi_\beta = \mu_\beta|_{\Sigma_0}$, where Σ_0 is a σ-algebra generated by Ω_0. The stochastic operator $\mathscr{T}$ of this Markov chain is called the transfer-matrix of the system. The operator $\mathscr{T}$ acts in the Hilbert space $\mathscr{H} = L_2(\Omega_0, \pi_\beta) \subset L_2(\Omega, \mu_\beta)$ of functions dependent only on the configuration $\sigma_0 \in \Omega_0$. The matrix elements of $\mathscr{T}$ could be written as:

$$(\mathscr{T} f, g) = \langle f(\sigma_1) \cdot \overline{g(\sigma_0)} \rangle_{\mu_\beta}, \qquad f, g \in \mathscr{H}. \tag{5}$$

We formulate now the main result of the paper. Let $m_k = \langle \sigma^k \rangle_\nu$ be the kth moment of σ, and the orthogonal complement in $\mathscr{H}$ to the subspace of constants $\mathscr{H}_0 = \{\mathbf{1}\} \subset \mathscr{H}$ is designated as $\mathscr{H}' = \mathscr{H} \ominus \mathscr{H}_0$. We consider two elements from $\mathscr{H}'$:

$$\sigma - m_1, \qquad \sigma^2 - m_2,$$

and define their second moments:

$$\begin{aligned} I_1 &= \langle (\sigma - m_1)^2 \rangle_\nu = m_2 - m_1^2, \\ I_2 &= \langle (\sigma^2 - m_2)^2 \rangle_\nu = m_4 - m_2^2, \\ I_{1,2} &= \langle (\sigma - m_1)(\sigma^2 - m_2) \rangle_\nu = m_3 - m_1 m_2. \end{aligned} \tag{6}$$

From our assumption (3) it easy follows that

$$I_1 > 0, \qquad I_2 > 0, \qquad I_1 I_2 - I_{1,2}^2 > 0$$

Let us denote by

$$d_1 = I_1, \qquad d_2 = I_2 - \frac{I_{1,2}^2}{I_1} > 0, \tag{7}$$

then the following theorem holds.

Theorem. Let $\beta > 0$ be small enough, and

$$d_1^2 \neq \tfrac{1}{2} d_2. \tag{8}$$

Then the space $\mathscr{H}'$ could be decomposed into direct sum of mutually orthogonal subspaces:

$$\mathscr{H}' = \mathscr{H}_1 \oplus \mathscr{H}_2^{(1)} \oplus \mathscr{H}_2^{(2)} \oplus \mathscr{H}_3, \tag{9}$$

which are invariant with respect to the transfer-matrix $\mathscr{T}$ and the unitary group of the space translations $\{U_s, s \in Z^d\}$.

The spectra of $\mathscr{T}$ on these subspaces are not overlapping and meet the following conditions:

$$\begin{aligned} &\sigma(\mathscr{T}|_{\mathscr{H}_1}) \subset (d_1 \beta - c_1 \beta^2, d_1 \beta + c_1 \beta^2), \\ &\sigma(\mathscr{T}|_{\mathscr{H}_2^{(1)}}) \subset (\tfrac{1}{2} d_2 \beta^2 - c_2 \beta^3, \tfrac{1}{2} d_2 \beta^2 + c_2 \beta^3), \\ &\sigma(\mathscr{T}|_{\mathscr{H}_2^{(2)}}) \subset (d_1^2 \beta^2 - c_3 \beta^3, d_1^2 \beta^2 + c_3 \beta^3), \\ &\sigma(\mathscr{T}|_{\mathscr{H}_3}) \subset (-c_4 \beta^3, c_4 \beta^3) \end{aligned} \tag{10}$$

with absolute constants $c_i > 0$, $i = 1, 2, 3, 4$.

The proof of the theorem is based on the general approach to the investigation of leading branches of the transfer-matrix spectrum for lattice systems, which has been developed in the books of refs. 3 and 10. Here we will essentially use results and constructions from these books.

Remark. We study here the ferromagnetic ($\beta > 0$) lattice spin systems. However the similar results hold also for any small β. In the general case we have the same decomposition as (9), but the corresponding spectrum branches (10) could be found on both sides of the origin.

2. PRELIMINARY CONSTRUCTIONS

We remind here shortly main steps from refs. 3 and 4, constructing the invariant subspaces of the transfer-matrix.

2.1. A Multiplicative Basis in $\mathscr{H}'$

We denote by $\{v_k, k=0, 1, 2,...\}$ a finite or countable orthonormal basis in the space $h = L_2(S, dv)$, which is a result of the orthogonalization in h of the monoms $1, \sigma, \sigma^2,...$. In particular, $v_0 = 1$ and $v_1 \neq 0$, $v_2 \neq 0$:

$$v_1 = \frac{\sigma - m_1}{\sqrt{d_1}}, \quad v_2 = \frac{\sigma^2 - m_2 - \frac{I_{1,2}}{I_1}(\sigma - m_1)}{\sqrt{d_2}}, \tag{11}$$

where the constants I_1, $I_{1,2}$, d_1, d_2 are defined above by (6) and (7). Then we could construct an orthonormal in $\mathscr{H}$ system of the functions $\{\tilde{v}_k^x = \tilde{v}_k^x(\sigma_0),\ k = 0, 1, 2,...,\ x \in Y_0\}$, using the same reasoning as in ref. 3. These functions appears to be a small perturbation of the functions $\{v_k\}$, and the following representation holds:

$$\tilde{v}_k^x(\sigma) = v_k(\sigma(x)) + w_k^x(\sigma), \qquad \tilde{v}_0^x = 1, \quad \sigma \in \Omega_0, \tag{12}$$

and

$$|w_k^x(\sigma)| < C\beta,$$

C is an absolute constant. Let $\Gamma = \{k(x), x \in Z^d\}$, $k(x) \geqslant 0$ are nonnegative integer-value functions with finite support (multi-indexes), then the functions

$$\psi_\Gamma = \prod_{x \in Z^d} \tilde{v}_{k(x)}^x, \qquad \operatorname{supp} \Gamma \neq \varnothing, \tag{13}$$

form an orthonormal basis in $\mathscr{H}'$.

2.2. Cluster Expansion for the Matrix Elements of $\mathscr{T}$ in the Basis $\{\psi_\Gamma\}$

Using (5), the structure (13) of the orthonormal basis and the general expression for moments of random variables by its semi-invariants (see, for example, ref. 10) we obtain the following representation for the matrix elements of the transfer-matrix in the basis $\{\psi_\Gamma\}$, $\operatorname{supp} \Gamma \neq \varnothing$:

$$(\mathscr{T}\psi_\Gamma, \psi_{\Gamma'}) = \langle \psi_\Gamma(\sigma_1) \cdot \overline{\psi_{\Gamma'}(\sigma_0)} \rangle_{\mu_\beta} = \sum_{(\Delta_1,..., \Delta_n) = (\Gamma, \Gamma')} \omega_{\Delta_1} \omega_{\Delta_2} \cdots \omega_{\Delta_n}. \tag{14}$$

Here $\Gamma = \{k(x), x \in Z^d\}$, $\Gamma' = \{k'(y), y \in Z^d\}$,

$$\omega_\Delta = \omega_{(\Gamma, \Gamma')} = \left\langle \prod'_{x \in Z^d} \tilde{v}^x_{k(x)}(\sigma_1), \prod'_{y \in Z^d} \tilde{v}^y_{k'(y)}(\sigma_0) \right\rangle_{\mu_\beta} \tag{15}$$

is a semi-invariant for the functions of the form (12). The summation in (14) is over all partitions $(\Delta_1, ..., \Delta_n)$ of the pair (Γ, Γ') on subpairs $\Delta_i = (\Gamma_i, \Gamma'_i)$, such that

$$\begin{aligned}
&(1) \quad \operatorname{supp} \Gamma_i \neq \varnothing, \quad \operatorname{supp} \Gamma'_i \neq \varnothing, \\
&(2) \quad \operatorname{supp} \Gamma_i \cap \operatorname{supp} \Gamma_j = \varnothing, \quad \operatorname{supp} \Gamma'_i \cap \operatorname{supp} \Gamma'_j = \varnothing, \quad i \neq j, \\
&(3) \quad \bigcup_i \operatorname{supp} \Gamma_i = \operatorname{supp} \Gamma, \quad \bigcup_i \operatorname{supp} \Gamma'_i = \operatorname{supp} \Gamma', \\
&(4) \quad \Gamma_i(x) = k(x), \quad x \in \operatorname{supp} \Gamma_i, \quad \Gamma'_i(y) = k'(y); \quad y \in \operatorname{supp} \Gamma'_i
\end{aligned} \tag{16}$$

2.3. Cluster Estimates

Let us consider a multi-index Γ with support on Y_1, $\Gamma = \{k(x), x \in Y_1\}$, and $\operatorname{supp} \Gamma' \subset Y_0$ as before: $\Gamma' = \{k'(y), y \in Y_0\}$. The semi-invariants (15) satisfy the following estimates:

$$|\omega_\Delta| < B(C\beta)^{\kappa(\Delta)}, \quad \Delta = (\Gamma, \Gamma'), \quad \operatorname{supp} \Gamma \subset Y_1, \quad \operatorname{supp} \Gamma' \subset Y_0 \tag{17}$$

with absolute constants B, C, and

$$\kappa(\Delta) = \min_{\Phi} \left(\sum_b \Phi(b) \right), \tag{18}$$

where $\Phi = \{\Phi(b)\}$ is a nonnegative integer-value function defined on the bonds b of the lattice Z^{d+1}. The function Φ meets the following conditions:

(1) $\operatorname{supp} \Phi$ is a connected sub-graph of the lattice Z^{d+1};

(2) for any point $x \in Y_1 \cup Y_0$

$$\sum_{b:\, x \in \partial b} \Phi(b) \geqslant \begin{cases} k(x), & x \in Y_1, \\ k'(x), & x \in Y_0, \end{cases} \tag{19}$$

with $\Gamma = \{k(x), x \in Y_1\}$, $\Gamma' = \{k'(x), x \in Y_0\}$. A set of vertices of the bond b is denoted by ∂b.

The representation (14) together with (17)–(19) imply the following estimate on the matrix elements of the operator $\mathcal{T}$ adapted to our case. Let us consider a function $\xi(\Delta)$ defined on the pairs $\Delta=(\Gamma, \Gamma')$, $\operatorname{supp} \Gamma \subset Y_1$, $\operatorname{supp} \Gamma' \subset Y_0$:

$$\xi(\Delta)=d(\Delta)+\frac{1}{2}\left[\sum_{x \in Y_1}(k(x)-n(x))_{+}+\sum_{x \in Y_0}(k'(x)-n(x))_{+}\right]^{ev} \tag{20}$$

Here $d(\Delta)$ is the length of the minimal connected subgraph τ on the lattice Z^{d+1}, such that

$$\operatorname{supp} \Gamma \cup \operatorname{supp} \Gamma' \subset \partial\tau,$$

and $n(x)$ is a degree of the point $x \in \partial\tau$, $n(x)=\#\{b \in \tau : x \in \partial b\}$, $(m)_{+}=\max\{0, m\}$, $[M]^{ev}=\begin{cases} M, & \text{if } M \text{ is even,} \\ M+1, & \text{if } M \text{ is odd.}\end{cases}$

We denote by

$$\xi(\Gamma, \Gamma')=\min_{(\Delta_1, \Delta_2, \ldots, \Delta_n)} \sum_{i=1}^{n} \xi(\Delta_i), \tag{21}$$

where the minimum is taken over all partitions (16) of the pair (Γ, Γ'). Then the following estimate holds:

$$|(\mathcal{T}\psi_{\Gamma}, \psi_{\Gamma'})|<B(C\beta)^{\xi(\Gamma, \Gamma')} \tag{22}$$

with absolute constants B, C.

2.4. The General Construction of Invariant Subspaces

Let a Hilbert space $\mathcal{H}$ is decomposed into a direct sum of orthogonal subspaces

$$\mathcal{H}=\mathcal{R}_1 \oplus \mathcal{R}_2 \tag{23}$$

Then the decomposition (23) implies the following matrix representation for any bounded self-adjoint operator L:

$$L=\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \tag{24}$$

with L_{ij}: $\mathcal{R}_j \to \mathcal{R}_i$, $i, j=1, 2$. We'll formulate below conditions, which guarantee the existence of a unique invariant subspace $\mathcal{H}_1 \subset \mathcal{H}$ of the operator L, "close" to the subspace $\mathscr{R}_1$ in the sense that the subspace $\mathscr{H}_1$ has the form of the graph

$$\mathscr{H}_1 = \{w\colon w = v + Sv,\ v \in \mathscr{R}_1\} \tag{25}$$

of an operator $\mathscr{S}\colon \mathscr{R}_1 \to \mathscr{R}_2$ with a small norm $\|S\|$.

Lemma 1. We suppose that there exists a bounded operator L_{11}^{-1}, and

$$\begin{aligned} &(1)\quad \|L_{22}\| \cdot \|L_{11}^{-1}\| = q < 1, \\ &(2)\quad \|L_{21}\| \cdot \|L_{11}^{-1}\| = \|L_{12}\| \cdot \|L_{11}^{-1}\| = \varepsilon < \frac{1-q}{2}. \end{aligned} \tag{26}$$

Then for $\delta = \frac{2\varepsilon}{1-q}$ there exists a unique invariant subspace $\mathscr{H}_1 \subset \mathscr{H}$ of the form (25) with $\|S\| < \delta$.

Proof. The condition of the invariance for $\mathscr{H}_1$ with respect to L could be rewritten using (24) and (25) by the following way:

$$S(L_{11} + L_{12}S) = L_{21} + L_{22}S$$

or

$$S = L_{21}L_{11}^{-1} + L_{22}SL_{11}^{-1} - SL_{12}SL_{11}^{-1} \equiv \mathscr{F}(S). \tag{27}$$

The right-hand side of Eq. (27) could be considered as a transformation

$$\mathscr{F}\colon \mathscr{L}(\mathscr{R}_1, \mathscr{R}_2) \to \mathscr{L}(\mathscr{R}_1, \mathscr{R}_2)$$

in the space of bounded operators from $\mathscr{R}_1$ to $\mathscr{R}_2$. It is easy to check that a ball in $\mathscr{L}(\mathscr{R}_1, \mathscr{R}_2)$ of the size $\delta = \frac{2\varepsilon}{1-q}$:

$$\mathscr{B}_\delta = \{S \in \mathscr{L}(\mathscr{R}_1, \mathscr{R}_2) : \|S\| < \delta\}$$

is invariant with respect to the transformation $\mathscr{F}$. In addition $\mathscr{F}$ is a contraction on $\mathscr{B}_\delta$:

$$\|\mathscr{F}(S_1) - \mathscr{F}(S_2)\| \leqslant k\|S_1 - S_2\| \quad \text{with } k < 1 \quad \text{for any } S_1, S_2 \in \mathscr{B}_\delta.$$

Consequently, Eq. (27) has a unique solution inside $\mathscr{B}_\delta$.

Corollary 1. If $\mathscr{U}$ is a unitary operator in $\mathscr{H}$ commuting with the operator L, such that $\mathscr{R}_1$ and $\mathscr{R}_2$ are invariant with respect to $\mathscr{U}$, then under conditions of Lemma 1 the subspace $\mathscr{H}_1$ (25) is also invariant with respect to $\mathscr{U}$.

This statement follows from the observation that if $S \in \mathscr{B}_\delta$ is the solution of (27), then the operator $\mathscr{U}S\mathscr{U}^{-1}$ also satisfies Eq. (27), and $\mathscr{U}S\mathscr{U}^{-1} \in \mathscr{B}_\delta$. Hence $\mathscr{U}S\mathscr{U}^{-1} = S$, i.e., $\mathscr{U}$ commutes with S, and the subspace $\mathscr{H}_1$ is invariant with respect to $\mathscr{U}$.

Corollary 2. If $\{v_\alpha, \alpha \in A\}$ is an orthonormal basis in $\mathscr{R}_1$, marked by elements of a set A, then functions

$$u_\alpha = \sum_{\alpha' \in A} (E_{\mathscr{R}_1} + S^*S)^{-1/2}_{\alpha, \alpha'} (v_{\alpha'} + \mathscr{S} v_{\alpha'}) \tag{28}$$

form an orthonormal basis in the invariant subspace $\mathscr{H}_1$. Here $E_{\mathscr{R}_1}$ is the identical operator in $\mathscr{R}_1$, S^*: $\mathscr{R}_2 \to \mathscr{R}_1$ is the conjugate operator to S, and we denote by $K_{\alpha, \alpha'}$ matrix elements of an operator K: $\mathscr{R}_1 \to \mathscr{R}_1$ in the basis $\{v_\alpha\}$.

Corollary 3. For any function $g = v + Sv \in \mathscr{H}_1$, $v \in \mathscr{R}_1$ we have

$$Lg = (L_{11} + L_{12}S)\, v + S(L_{11} + L_{12}S)\, v, \tag{29}$$

so that the operator $L|_{\mathscr{H}_1}$ is similar to the operator $(L_{11} + L_{12}S)$ acting in $\mathscr{R}_1$:

$$(E_{\mathscr{R}_1} + S)^{-1}\, L(E_{\mathscr{R}_1} + S) = L_{11} + L_{12}S$$

where $(E_{\mathscr{R}_1} + S)^{-1}$: $\mathscr{H}_1 \to \mathscr{R}_1$ is the inverse operator to $(E_{\mathscr{R}_1} + S)$: $\mathscr{R}_1 \to \mathscr{H}_1$.

Corollary 4. The orthogonal complement to $\mathscr{H}_1$

$$\mathscr{H}_1^\perp = \mathscr{H} \ominus \mathscr{H}_1$$

is also invariant with respect to L. Similar reasoning shows that the subspace $\mathscr{H}_1^\perp$ is the same as the graph of the operator $(-S^*)$: $\mathscr{R}_2 \to \mathscr{R}_1$

$$\mathscr{H}_1^\perp = \{f \colon f = u - S^*u,\ u \in \mathscr{R}_2\} \tag{30}$$

By analogy with (28), functions

$$w_\beta = \sum_{\beta' \in B} (E_{\mathscr{R}_2} + SS^*)^{-1/2}_{\beta, \beta'} (u_{\beta'} - S^* u_{\beta'}) \tag{31}$$

form an orthonormal basis in $\mathscr{H}_1^{\perp}$, where $\{u_\beta, \beta \in B\}$ is an orthonormal basis in $\mathscr{R}_2$.

Since for any function $f = u - S^* u \in \mathscr{H}_1^{\perp}$, $u \in \mathscr{R}_2$ we have

$$Lf = (L_{22} - L_{21} S^*)\, u - S^*(L_{22} - L_{21} S^*)\, u, \tag{32}$$

then the operator $L|_{\mathscr{H}_1^{\perp}}$ is similar to the operator $(L_{22} - L_{21} S^*)$ in $\mathscr{R}_2$.

3. THE INVARIANT SUBSPACE $\mathscr{H}_1$

In this section we start to prove the main theorem using the above constructions. The first step of the proof is to find the invariant subspace $\mathscr{H}_1 \subset \mathscr{H}'$. According to the reasoning of the previous section, we consider a decomposition of our Hilbert space $\mathscr{H}'$ on two orthogonal subspaces:

$$\mathscr{H}' = R_1 \oplus R_2, \qquad R_1 \perp R_2. \tag{33}$$

Here R_1 is the linear span of the functions of the form (12) with $k = 1$:

$$R_1 = \{\psi_\Gamma(\sigma_0), |\Gamma| = 1\} \equiv \{\tilde{v}_1^x(\sigma_0), x \in Y_0\} \tag{34}$$

(so that the functions $\tilde{v}_1^x$, $x \in Y_0$ form the basis in R_1), and

$$R_2 = \{\psi_\Gamma(\sigma_0), |\Gamma| \geqslant 2\}$$

with

$$|\Gamma| = \sum_x k(x) \qquad \text{for} \quad \Gamma = \{k(x), x \in Z^d\}.$$

The decomposition (33) implies the matrix representation for the transfer-matrix

$$\mathscr{T} = \begin{pmatrix} \mathscr{T}_{11} & \mathscr{T}_{12} \\ \mathscr{T}_{21} & \mathscr{T}_{22} \end{pmatrix} \tag{35}$$

with $\mathscr{T}_{i,j}\colon R_j \to R_i$, $i, j = 1, 2$. Then the following estimates hold.

Lemma 2. For small enough β the operator $\mathscr{T}_{11}^{-1}$ exists, and

$$\|\mathscr{T}_{11}^{-1}\| \leqslant \frac{2}{d_1 \beta}, \tag{36}$$

where d_1 was defined by (7).

In addition

$$\max\{\|\mathcal{T}_{12}\|, \|\mathcal{T}_{21}\|, \|\mathcal{T}_{22}\|\} \leqslant C_1\beta^2 \tag{37}$$

with an absolute constant C_1.

The proof of Lemma 2 is given in Appendix A.

Applying now Lemma 1 to the decompositions (33) and (35) together with the estimates (36) and (37) we construct an invariant with respect to $\mathcal{T}$ subspace $\mathcal{H}_1 \subset \mathcal{H}'$ of the form (25). The corresponding operator $S: R_1 \to R_2$ meets the following estimate:

$$\|S\| < C\beta, \tag{38}$$

with an absolute constant C.

Since the representations of the space translations $\{\mathcal{U}_s, s \in Z^d\}$ form the group of unitary operators commuting with $\mathcal{T}$, and the subspaces R_1, R_2 in (33) are invariant with respect to $\mathcal{U}_s$, then Corollary 1 implies that $\mathcal{H}_1$ is invariant with respect to the unitary group $\{\mathcal{U}_s, s \in Z^d\}$. In addition for any $x, s \in Z^d$ we have

$$\mathcal{U}_s u_x = u_{x+s} \tag{39}$$

where $\{u_x, x \in Y_0\}$ is the orthonormal basis in $\mathcal{H}_1$ constructed by (28) from the functions $\{\tilde{v}_1^x\}$. The representation (39) means that the subspace $\mathcal{H}_1$ is cyclic with respect to the unitary group of the space translations $\{\mathcal{U}_s, s \in Z^d\}$, so that $\mathcal{H}_1$ has the so-called "one-particle" structure, and it could be called as a quasi-particle states space.

Corollary 3 implies that the spectra of the operators $\mathcal{T}_1 = \mathcal{T}|_{\mathcal{H}_1}$ and $(\mathcal{T}_{11} + \mathcal{T}_{12}S)$ in R_1 are the same. Using the estimates (36) and (37) and the results (59) from the proof of Lemma 2 (see Appendix A) one could easy obtain that the spectrum of $\mathcal{T}_1$ lies in a $C\beta^2$-neighborhood of the point $d_1\beta$:

$$\sigma(\mathcal{T}|_{\mathcal{H}_1}) \subset (d_1\beta - C\beta^2, d_1\beta + C\beta^2)$$

with a constant C.

Finally using the construction (30) of the orthogonal complement $\mathcal{H}_1^\perp$ and the formulas (31), (32), (37), (38), (59) we have, that the spectrum of the operator $\mathcal{T}|_{\mathcal{H}_1^\perp}$ belongs to a $\tilde{C}_1\beta^2$-neighborhood of the origin:

$$\sigma(\mathcal{T}|_{\mathcal{H}_1^\perp}) \subset (-\tilde{C}_1\beta^2, \tilde{C}_1\beta^2)$$

with a constant $\tilde{C}_1 > 0$. Thus all statements of the theorem concerning to the first invariant subspace are proved.

4. THE DECOMPOSITION OF THE SUBSPACE $\mathcal{H}_1^{\perp}$

In this section we get first a decomposition of the space $\mathcal{H}_1^{\perp}$ into two subspaces invariant with respect to $\mathcal{T}$ and $\{\mathcal{U}_s\}$:

$$\mathcal{H}_1^{\perp} = \mathcal{H}_2 \oplus \mathcal{H}_3 \tag{40}$$

with $\sigma(\mathcal{T}|_{\mathcal{H}_2}) \sim \beta^2$ and $\sigma(\mathcal{T}|_{\mathcal{H}_3}) \sim \beta^3$. Then we find that

$$\mathcal{H}_2 = \mathcal{H}_2^{(1)} \oplus \mathcal{H}_2^{(2)} \tag{41}$$

and prove that the spectra $\mathcal{T}$ on $\mathcal{H}_2^{(1)}$ and $\mathcal{H}_2^{(2)}$ respectively are not overlapping. Thus the decomposition (9) will be constructed.

4.1. Decomposition (40)

By Corollary 4 the functions

$$\hat{\psi}_\Gamma(\sigma_0) = \sum_{\Gamma' : |\Gamma'| \geqslant 2} (E_{R_2} + SS^*)^{-1/2}_{\Gamma, \Gamma'} (\psi_{\Gamma'}(\sigma_0) - S^* \psi_{\Gamma'}(\sigma_0)), \tag{42}$$

marked by multi-indices Γ with $|\Gamma| \geqslant 2$, form an orthonormal basis in $\mathcal{H}_1^{\perp}$. Let us consider a decomposition of $\mathcal{H}_1^{\perp}$ into an orthogonal sum of subspaces, invariant with respect to the unitary group of the space translations $\{\mathcal{U}_s, s \in Z^d\}$:

$$\mathcal{H}_1^{\perp} = \hat{R}_1 \oplus \hat{R}_2, \tag{43}$$

where

$$\hat{R}_1 = \{\hat{\psi}_\Gamma(\sigma_0), |\Gamma| = 2\}, \qquad \hat{R}_2 = \{\hat{\psi}_\Gamma(\sigma_0), |\Gamma| \geqslant 3\}.$$

The decomposition (43) generates the matrix representation for the restriction of $\mathcal{T}$ on $\mathcal{H}_1^{\perp}$:

$$\mathcal{T}|_{\mathcal{H}_1^{\perp}} = \begin{pmatrix} \hat{\mathcal{T}}_{11} & \hat{\mathcal{T}}_{12} \\ \hat{\mathcal{T}}_{21} & \hat{\mathcal{T}}_{22} \end{pmatrix} \tag{44}$$

Lemma 3. For small enough β the operator $\hat{\mathcal{T}}_{11}^{-1}$ exists and

$$\|\hat{\mathcal{T}}_{11}^{-1}\| \leqslant \frac{C_1}{\beta^2}. \tag{45}$$

In addition

$$\max\{\|\widehat{\mathscr{T}}_{12}\|, \|\widehat{\mathscr{T}}_{21}\|, \|\widehat{\mathscr{T}}_{22}\|\} \leqslant C_2\beta^3 \tag{46}$$

with constants C_1, C_2.

The proof is given in Appendix B.

Now we are again under conditions of Lemma 1 with $q=\mathcal{O}(\beta)$, $\varepsilon=\mathcal{O}(\beta)$. Consequently, the operator $\hat{S}: \hat{R}_1 \to \hat{R}_2$ with a small norm $\|\hat{S}\| < C\beta$ exists, and the unique invariant subspace $\mathscr{H}_2$ of the form (25) could be constructed with the help of $\hat{S}$.

By Corollary 3 the operator $\mathscr{T}_2 = \mathscr{T}|_{\mathscr{H}_2}$ has the same spectrum as the operator $(\widehat{\mathscr{T}}_{11} + \widehat{\mathscr{T}}_{12}\hat{S})$ on $\hat{R}_1$. Using the estimates (45) and (46) and the representations (66) and (67) for $\widehat{\mathscr{T}}_{11}$ (see Appendix B) one could easy see that the spectrum of $\mathscr{T}_2$ is concentrated in $C\beta^3$-neighborhoods of the points $d_1^2\beta^2$ and $\frac{1}{2}d_2\beta^2$.

We denote by $\mathscr{H}_3$ the orthogonal complement to $\mathscr{H}_2$:

$$\mathscr{H}_3 = \mathscr{H}_1^{\perp} \ominus \mathscr{H}_2 = (\mathscr{H}' \ominus \mathscr{H}_1) \ominus \mathscr{H}_2.$$

Using the results of Corollary 4 we get that the spectrum of the operator $\mathscr{T}|_{\mathscr{H}_3}$ is the same as the spectrum of the operator $(\widehat{\mathscr{T}}_{22} - \widehat{\mathscr{T}}_{21}\hat{S}^*)$ on $\hat{R}_2$. The bounds (45), (46) imply that

$$\sigma(\mathscr{T}|_{\mathscr{H}_3}) \subset (-\tilde{C}_2\beta^3, \tilde{C}_2\beta^3)$$

with a constant $\tilde{C}_2 > 0$.

4.2. Decomposition (41)

Using the construction of $\mathscr{H}_2$ and Corollary 2 we have, that the functions

$$\tilde{\psi}_\Gamma(\sigma_0) = \sum_{\Gamma': |\Gamma'|=2} (E_{\hat{R}_1} + \hat{S}^*\hat{S})^{-1/2}_{\Gamma,\Gamma'} (\hat{\psi}_{\Gamma'}(\sigma_0) + \hat{S}^*\hat{\psi}_{\Gamma'}(\sigma_0)), \tag{47}$$

marked by multi-indices Γ with $|\Gamma|=2$, form an orthonormal basis in $\mathscr{H}_2$. Let us consider a decomposition of $\mathscr{H}_2$ on two orthogonal subspaces:

$$\mathscr{H}_2 = \tilde{R}_1 \oplus \tilde{R}_2, \tag{48}$$

where $\tilde{R}_1$ is the linear span of the basis functions (47), marked by Γ with $\operatorname{supp} \Gamma = \{x\} \subset Y_0$ and $k(x)=2$; $\tilde{R}_2$ is the linear span of the basis functions

(47), marked by Γ with supp $\Gamma = \{x, y\} \subset Y_0$, $y \neq x$ and $k(x) = k(y) = 1$. The decomposition (48) implies as above the matrix representation for $\mathcal{T}|_{\mathcal{H}_2}$:

$$\mathcal{T}|_{\mathcal{H}_2} = \begin{pmatrix} \widetilde{\mathcal{T}}_{11} & \widetilde{\mathcal{T}}_{12} \\ \widetilde{\mathcal{T}}_{21} & \widetilde{\mathcal{T}}_{22} \end{pmatrix}$$

Lemma 4. For small enough β and $d_1^2 \neq \frac{1}{2} d_2$ the following representations hold:

$$\widetilde{\mathcal{T}}_{11} = \tfrac{1}{2} d_2 \beta^2 E_{\widetilde{R}_1} + \widetilde{L}_1, \qquad \widetilde{\mathcal{T}}_{22} = d_1^2 \beta^2 E_{\widetilde{R}_2} + \widetilde{L}_2. \tag{49}$$

In addition

$$\max\{\|\widetilde{L}_1\|, \|\widetilde{L}_2\|, \|\widetilde{\mathcal{T}}_{12}\|, \|\widetilde{\mathcal{T}}_{21}\|\} \leqslant C\beta^3 \tag{50}$$

Here the constants d_1, d_2 were defined by (7), C is an absolute constant.

The proof is given in Appendix C.

Let us assume $d_1^2 < \frac{1}{2} d_2$. The case $d_1^2 > \frac{1}{2} d_2$ could be considered by the similar way. The representations (49) and the estimate (50) imply the existence of the inverse operators $\widetilde{\mathcal{T}}_{11}^{-1}$, $\widetilde{\mathcal{T}}_{22}^{-1}$ with the following norms:

$$\|\widetilde{\mathcal{T}}_{11}^{-1}\| \leqslant \frac{2}{d_2 \beta^2}(1 + C_1 \beta), \qquad \|\widetilde{\mathcal{T}}_{22}^{-1}\| \leqslant \frac{1}{d_1^2 \beta^2}(1 + C_2 \beta) \tag{51}$$

with absolute constants C_j, $j = 1, 2$. From (49)–(51) it follows that for small enough β the conditions of Lemma 1 are fulfilled:

$$\|\widetilde{\mathcal{T}}_{22}\| \cdot \|\widetilde{\mathcal{T}}_{11}^{-1}\| = q < \frac{2d_1^2}{d_2} + \mathcal{O}(\beta) < 1 \qquad \text{and} \qquad \varepsilon < C\beta$$

with a constant C. Consequently we can find as above the operator $\widetilde{S}$: $\widetilde{R}_1 \to \widetilde{R}_2$ and the subspace $\mathcal{H}_2^{(1)}$ of the form (25), "close" to $\widetilde{R}_1$ and invariant with respect to the operators $\mathcal{T}$ and $\{\mathcal{U}_s\}$. By analogy with above reasoning the spectrum of the operator $\mathcal{T}|_{\mathcal{H}_2^{(1)}}$ (which is the same as the spectrum of the operator $(\widetilde{\mathcal{T}}_{11} + \widetilde{\mathcal{T}}_{12}\widetilde{S})$ in $\widetilde{R}_1$) occupies a $C_2\beta^3$-neighborhood of the point $\frac{1}{2} d_2 \beta^2$:

$$\sigma(\mathcal{T}|_{\mathcal{H}_2^{(1)}}) \subset (\tfrac{1}{2} d_2 \beta^2 - C_2 \beta^3, \tfrac{1}{2} d_2 \beta^2 + C_2 \beta^3). \tag{52}$$

The orthogonal complement to $\mathcal{H}_2^{(1)}$:

$$(\mathcal{H}_2^{(1)})^{\perp} = \mathcal{H}_2 \ominus \mathcal{H}_2^{(1)} \equiv \mathcal{H}_2^{(2)} = \widetilde{R}_2 \oplus (-\widetilde{S}^* \widetilde{R}_2), \qquad \widetilde{S}^*: \widetilde{R}_2 \to \widetilde{R}_1$$

is also invariant with respect to $\mathscr{T}$ and $\{\mathscr{U}_s\}$. Using as above Corollary 4 with (49) and (50) we obtain that the spectrum of the operator $\mathscr{T}|_{\mathscr{H}_2^{(2)}}$ is in a $C_3\beta^3$-neighborhood of the point $d_1^2\beta^2$:

$$\sigma(\mathscr{T}|_{\mathscr{H}_2^{(2)}}) \subset (d_1^2\beta^2 - C_3\beta^3,\, d_1^2\beta^2 + C_3\beta^3), \tag{53}$$

so that for small enough β the spectra (52) and (53) are not overlapping.

The theorem is proven.

APPENDIX A. THE PROOF OF LEMMA 2

In what follows we will exploit a general formula giving the expression for moments by way of semi-invariants of the free field (see, for instance, refs. 4 and 10). For small enough β and any bounded function F on Ω we have:

$$\langle F\rangle_{\mu_\beta} = \sum_{m=0}^{\infty} \frac{\beta^m}{m!} \sum_{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle,\ldots, \langle x_m, y_m\rangle} \langle F, \sigma_{x_1}\sigma_{y_1},\ldots, \sigma_{x_m}\sigma_{y_m}\rangle_{\mu_0}. \tag{54}$$

Let us consider the following cases.

1. If $\|\Gamma\| = \|\Gamma'\| = 1$ with supp $\Gamma = \{x\} \in Y_1$, supp $\Gamma' = \{y\} \in Y_0$, then in the designations of the formulas (21) and (22) we have

$$\xi(\Gamma, \Gamma') \geqslant d(\bar{\Delta}) = \|x - y\| \geqslant 1,$$

and

$$(\mathscr{T}\psi_\Gamma, \psi_{\Gamma'}) \leqslant C_1\beta(C\beta)^{\|x-y\|-1} \tag{55}$$

with absolute constants C_1, C. In addition using the formulas (11), (12), and (54) we get

$$\begin{aligned}(\mathscr{T}\tilde{v}_1^x(\sigma_0), \tilde{v}_1^x(\sigma_0)) &= \langle \tilde{v}_1^x(\sigma_1)\cdot\overline{\tilde{v}_1^x(\sigma_0)}\rangle_{\mu_\beta}\\ &= \beta\langle v_1(\sigma_1(x))\cdot\overline{v_1(\sigma_0(x))}, \sigma_0(x)\,\sigma_1(x)\rangle_{\mu_0} + \mathscr{O}(\beta^3)\\ &= \beta\langle v_1(\sigma_0(x))\cdot\sigma_0(x)\rangle_{\mu_0}^2 + \mathscr{O}(\beta^2)\\ &= \beta(m_2 - m_1) + \mathscr{O}(\beta^2) = d_1\beta + \mathscr{O}(\beta^2).\end{aligned} \tag{56}$$

2. If $\|\Gamma\| = 1$, $\|\Gamma'\| \geqslant 2$, then $\xi(\Gamma, \Gamma') \geqslant 2$, and

$$(\mathscr{T}\psi_\Gamma, \psi_{\Gamma'}) \leqslant C_2\beta^2(C\beta)^{\xi(\Gamma,\Gamma')-2} \tag{57}$$

with constants C_2, C.

3. If $\|\Gamma\| \geqslant 2$, $\|\Gamma'\| \geqslant 2$, then $\xi(\Gamma, \Gamma') \geqslant 2$, and

$$(\mathscr{T}\psi_{\Gamma}, \psi_{\Gamma'}) \leqslant C_3 \beta^2 (C\beta)^{\xi(\Gamma, \Gamma')-2} \tag{58}$$

with constants C_3, C.

To study the operator $\mathscr{T}_{11}^{-1}$ we notice that (55) and (56) imply the following representation for the operator $\mathscr{T}_{11}$:

$$\mathscr{T}_{11} = d_1 \beta E_{R_1} + \tilde{\mathscr{T}}_{11}, \qquad \|\tilde{\mathscr{T}}_{11}\| \leqslant C_0 \beta^2, \tag{59}$$

where E_{R_1} is the identity operator in R_1. Hence for small enough β the operator $\mathscr{T}_{11}^{-1}$ exists and it could be written as

$$\mathscr{T}_{11}^{-1} = \frac{1}{d_1 \beta} (E_{R_1} + T_1) \tag{60}$$

with $\|T_1\| \leqslant C_1 \beta$, C_1 is an absolute constant. Now (60) implies the estimate (36).

Finally using the same reasoning as in refs. 3 and 4, we could show that the estimates (57) and (58) imply the convergence of the corresponding series providing the estimates (37).

Lemma 2 is proven.

APPENDIX B. THE PROOF OF LEMMA 3

The proof is analogous to the proof of Lemma 2. It is based on the bound (22) (under notations (20) and (21)) for matrix elements of $\mathscr{T}$ and the reasoning from refs. 3 and 4. First we consider matrix elements of the form

$$(\hat{\mathscr{T}}_{11} \hat{\psi}_{\Gamma}, \hat{\psi}_{\Gamma'}) = (\mathscr{T} \hat{\psi}_{\Gamma}, \hat{\psi}_{\Gamma'}) \qquad \text{with} \quad |\Gamma| = |\Gamma'| = 2.$$

The set of all multi-indices Γ with $|\Gamma| = 2$ is divided into two families:

$$\begin{aligned} \gamma_1 &= \{|\Gamma_1| = 2\text{: supp } \Gamma_1 = \{x\},\, x \subset Y_0,\, k(x) = 2\}, \\ \gamma_2 &= \{|\Gamma_2| = 2\text{: supp } \Gamma_2 = \{x, y\} \subset Y_0,\, x \neq y,\, k(x) = k(y) = 1\}. \end{aligned} \tag{61}$$

Let us consider the following cases.

1. If $\Gamma \in \gamma_1$ then using (54), (42), (11)–(13) and the estimates on the operators S, S^* (see Section 3) we have:

$$(\mathscr{T}\hat{\psi}_\Gamma, \hat{\psi}_\Gamma) = \langle \hat{\psi}_\Gamma(\sigma_1)\cdot\hat{\psi}_\Gamma(\sigma_0)\rangle_{\mu_\beta} = \langle \tilde{v}_2^x(\sigma_1)\cdot\overline{\tilde{v}_2^x(\sigma_0)}\rangle_{\mu_\beta} + \mathcal{O}(\beta^3)$$
$$= \frac{\beta^2}{2}\langle v_2^x(\sigma_1)\cdot\overline{v_2^x(\sigma_0)}, \sigma_0(x)\,\sigma_1(x), \sigma_0(x)\,\sigma_1(x)\rangle_{\mu_0} + O(\beta^3)$$
$$= \frac{\beta^2}{2}\langle v_2^x(\sigma_0)\cdot\sigma_0^2(x)\rangle_{\mu_0}^2 + O(\beta^3) = \frac{d_2}{2}\beta^2 + \mathcal{O}(\beta^3). \tag{62}$$

2. If $\Gamma, \Gamma' \in \gamma_1, \Gamma \neq \Gamma'$, then $\xi(\Gamma, \Gamma') \geqslant 4$ and the cluster estimate (22) implies that

$$(\mathscr{T}\hat{\psi}_\Gamma, \hat{\psi}_{\Gamma'}) \leqslant L(C\beta)^4. \tag{63}$$

3. If $\Gamma \in \gamma_2$, then as above in the case 1 we have:

$$(\mathscr{T}\hat{\psi}_\Gamma, \hat{\psi}_\Gamma) = \langle \hat{\psi}_\Gamma(\sigma_1)\cdot\hat{\psi}_\Gamma(\sigma_0)\rangle_{\mu_\beta}$$
$$= \langle \tilde{v}_1^x(\sigma_1)\tilde{v}_1^y(\sigma_1)\,\overline{\tilde{v}_1^x(\sigma_0)\tilde{v}_1^y(\sigma_0)}\rangle_{\mu_\beta} + \mathcal{O}(\beta^3)$$
$$= \beta^2\langle v_1^x(\sigma_1)\,v_1^y(\sigma_1)\,\overline{v_1^x(\sigma_0)v_1^y(\sigma_0)}, \sigma_0(x)\,\sigma_1(x), \sigma_0(y)\,\sigma_1(y)\rangle_{\mu_0}$$
$$+ O(\beta^3)$$
$$= \beta^2\langle v_1^x(\sigma_0)\cdot\sigma_0(x)\rangle_{\mu_0}^4 + O(\beta^3) = d_1^2\beta^2 + \mathcal{O}(\beta^3). \tag{64}$$

4. If $\Gamma, \Gamma' \in \gamma_2, \Gamma \neq \Gamma'$ or $\Gamma \in \gamma_1, \Gamma' \in \gamma_2$, then the cluster estimate (22) implies

$$(\mathscr{T}\hat{\psi}_\Gamma, \hat{\psi}_{\Gamma'}) \leqslant B_1(C\beta)^3. \tag{65}$$

Taking into account the expressions (20) and (21) for $\xi(\Gamma, \Gamma')$ we obtain from (62)–(65) the following representation for $\hat{\mathscr{T}}_{11}$:

$$(\hat{\mathscr{T}}_{11})_{\Gamma,\Gamma'} = a(\Gamma)\,\delta_{\Gamma,\Gamma'} + M_{\Gamma,\Gamma'}, \qquad |\Gamma| = |\Gamma'| = 2, \tag{66}$$

where

$$a(\Gamma) = \begin{cases} \dfrac{d_2}{2}\beta^2, & \text{if} \quad \Gamma \in \gamma_1, \\ d_1^2\beta^2, & \text{if} \quad \Gamma \in \gamma_2, \end{cases} \tag{67}$$

and M is an operator in $\hat{R}_1$ with a small norm: $\|M\| \leqslant C\beta^3$, C is a constant. Consequently the operator $\hat{\mathscr{T}}_{11}^{-1}$ exists and

$$\|\hat{\mathscr{T}}_{11}^{-1}\| \leqslant \frac{(1+\tilde{C}_1\beta)}{\beta^2}\max\left\{\frac{2}{d_2}, \frac{1}{d_1^2}\right\}$$

with a constant $\tilde{C}_1$. The estimate (45) is proven.

To obtain the bound (46) we use the representation (42) for the functions $\hat{\psi}_{\Gamma}$, the estimates on the norms $\|S\|$, $\|S^*\|$ and the cluster estimate (22) on the matrix elements $(\mathcal{T}\hat{\psi}_{\Gamma}, \hat{\psi}_{\Gamma'})$, when $|\Gamma|=2$, $|\Gamma'|\geqslant 3$ or $|\Gamma|\geqslant 3$, $|\Gamma'|\geqslant 3$. We have in both cases

$$\xi(\Gamma, \Gamma')\geqslant 3,$$

so that finally the estimate (46) follows from (20) and (21).

Lemma 3 is proven.

APPENDIX C. THE PROOF OF LEMMA 4

Using the representation (47) and the estimates on the norms of the operators $\hat{S}$, $\hat{S}^*$ it is easy to see, that the functions $\tilde{\psi}_{\Gamma}$, $|\Gamma|=2$, are a "small" perturbation of the corresponding functions $\hat{\psi}_{\Gamma}$, $|\Gamma|=2$. Consequently the results (62)–(65) of Appendix B imply that if $\Gamma\in\gamma_1$, then

$$(\tilde{\mathcal{T}}_{11}\tilde{\psi}_{\Gamma}, \tilde{\psi}_{\Gamma}) = (\mathcal{T}\hat{\psi}_{\Gamma}, \hat{\psi}_{\Gamma}) + \mathcal{O}(\beta^3) = \tfrac{1}{2}\, d_2\beta^2 + \mathcal{O}(\beta^3); \tag{68}$$

if $\Gamma\in\gamma_2$, then

$$(\tilde{\mathcal{T}}_{22}\tilde{\psi}_{\Gamma}, \tilde{\psi}_{\Gamma}) = (\mathcal{T}\hat{\psi}_{\Gamma}, \hat{\psi}_{\Gamma}) + \mathcal{O}(\beta^3) = d_1^2\beta^2 + \mathcal{O}(\beta^3); \tag{69}$$

if $\Gamma\neq\Gamma'$, then

$$(\mathcal{T}\tilde{\psi}_{\Gamma}, \tilde{\psi}_{\Gamma'}) \leqslant B(C\beta)^{\xi(\Gamma,\Gamma')} \qquad \text{with} \quad \xi(\Gamma, \Gamma')\geqslant 3, \tag{70}$$

B, C are absolute constants.

Now the representations (49) follows from (68)–(70). The estimates (50) could be proven by the similar manner as above in Appendix B.

ACKNOWLEDGMENTS

The research was partially supported by DFG Grant 436 RUS 113/485/1, CRDF project RM1-2085, and Russian Foundation for Basic Research, Grant 99-01-00284 and Grant 00-01-00271. We are grateful to referee for useful remarks and references.

REFERENCES

1. R. S. Schor and M. O'Carroll, Transfer matrix spectrum and bound states for the lattice classical ferromagnetic spin systems at high temperature, *J. Stat. Phys.* **99**:1207–1223. Decay of the Bethe–Salpeter kernel for lattice classical ferromagnetic spin systems, *J. Stat. Phys.* **99**:1265–1279 (2000).

2. R. S. Schor and M. O'Carroll, Excitations for lattice classical ferromagnetic classical spin systems at high temperature: Noneven single-spin distributions, *Phys. Rev. E* **61**: 6156–6164 (2000).
3. V. A. Malyshev and R. A. Minlos, *Linear Infinite-Particle Operators* (American Mathematical Society Publ., 1995).
4. V. A. Malyshev and R. A. Minlos, Invariant subspaces of clustering operators. I., *J. Stat. Phys.* **21**:231–242 (1979); II., *Commun. Math. Phys.* **82**:211–226 (1981).
5. S. Mamatov and R. A. Minlos, Bound states of a two-particle cluster operator, *Teor. and Matem. Physika* **79**:1–163 (1989).
6. J. Abdullaev and R. A. Minlos, An extension of the Ising model, *Adv. Sov. Math.* **20**:1–20 (1994).
7. J. Glimm, A. Jaffe, and T. Spencer, The particle structure of the weakly coupled $\mathscr{P}(\varphi)_2$ model and other applications of high temperature expansions, in *Constructive Quantum Field Theory*, Springer Lecture Notes in Physics, Vol. 25 (Berlin, Heidelberg, New York, 1973).
8. T. Spencer and F. Zirilli, Scattering states and bound states in $\lambda\mathscr{P}(\varphi)_2$, *Comm. Math. Phys.* **49** (1976).
9. H. Koch, Irreducible kernels and bound states in $\lambda\mathscr{P}(\varphi)_2$ models, *Ann. Inst. Henri Poincare* **31**:173–234 (1979).
10. V. A. Malyshev and R. A. Minlos, *Gibbs Random Fields* (Kluwer Academic, 1991).
11. R. A. Minlos, Invariant k-particle subspaces of the Hamiltonian of chiral lattice fields, *Acta Appl. Math.* **22**:55–75 (1991).